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On Weierstrass 7-semigroups

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§1. Introduction.

Let $\mathbb{N}$ be the additive semigroup of non-negative integers. A subsemigroup H of $\mathbb{N}$ is called a *numerical semigroup* if the complement $\mathbb{N}\backslash H$ of H in $\mathbb{N}$ is a finite set. For any positive integer n a numerical semigroup H is called an *n-semigroup* if H starts with n, i.e., the minimum positive integer in H is n. For a non-singular complete irreducible curve C over an algebraically closed field k of characteristic 0 (which is called a *curve* in this paper) and its point P we set

$$H(P) = \{n \in \mathbb{N} | \exists \text{ a rational function } f \text{ on } C \text{ with } (f)_\infty = nP\}.$$

A numerical semigroup is *Weierstrass* if there exists a curve C with its point P such that $H(P) = H$. We are interested in the following problem:

Problem 1. Is every n-semigroup Weierstrass ?

We have the following positive results:

Fact 2. *For $n \leqq 5$ every n-semigroup is Weierstrass.* (For $n = 2$, classical, for $n = 3$, see [8] and for $n = 4, 5$, see [4] ,[5] respectively.)

But we know the negative result as follows:

Fact 3. *For any $n \geqq 13$, there exists a non-Weierstrass n-semigroup.* (For $n = 13$, [1] and for $n \geqq 14$ see, for example, [7].)

Thus, we have the following problem:

Problem 4. For $6 \leqq n \leqq 12$, is every n-semigroup Weierstrass or is there a non-Weierstrass n-semigroup ?

In this paper we are devoted to the study of 7-semigroups. In Section 2 we determine the 7-semigroups which are the semigroups $H(P)$ of ramification points P on cyclic coverings of the projective line $\mathbb{P}^1$ with degree 7. In Section 3 we divide the Weierstrass 7-semigroups generated by 4 elements into 31 cases and investigate whether such a 7-semigroup is of toric type in each case where a numerical semigroup is said to be *of toric type* if roughly speaking, the monomial curve associated to the numerical semigroup is a specialization of some affine toric variety, because we know that a numerical semigroup of toric type is Weierstrass ([4]).

§2. Cyclic 7-semigroups.

An n-semigroup is said to be *cyclic* if it is the semigroup $H(P)$ for some totally ramification point P on a cyclic covering of the projective line $\mathbb{P}^1$ with degree n. In this section we describe a necessary and sufficient condition on a 7-semigroup to be cyclc. Moreover, some non-cyclc Weierstrass 7-semigroups are given. We use the following notation: For an n-semigroup H we set

$$S(H) = \{n, s_1, \ldots, s_{n-1}\}$$

where $s_i = \mathrm{Min}\{h \in H | h \equiv i \bmod n\}$. We have the following necessary condition on an n-semigroup to be cyclc if n is prime.

Fact 5 ([9]). *Let p be a prime number. If H is a cyclic p-semigroup with*

$$S(H) = \{p, s_1, \ldots, s_{p-1}\},$$

then

$$s_i + s_{p-i} = s_j + s_{p-j}, \ \textit{all}\ i, j.$$

We had already obtained an answer to the converse problem of the above statement.

Fact 6. i) *For a prime number $p \leqq 7$, the converse of Fact 5 is true* (See [9]).
ii) *For any prime number $p \geqq 11$, the converse of Fact 5 is false* (See [3]).

By Fact 6 i) we get the following:

Proposition 7. *Let H be a 7-semigroup with*

$$S(H) = \{7, s_1, \ldots, s_6\}.$$

Assume that

$$s_1 + s_6 = s_2 + s_5 = s_3 + s_4.$$

Then H is cyclic, hence Weierstrass.

For any positive integers $b_0, \ldots, b_m$, $< b_0, \ldots, b_m >$ denotes the semigroup generated by $b_0, \ldots, b_m$. We give examples of cyclic 7-semigroups.

Example 8. (1) Let $H =< 7, 8, 10, 12 >$. Then $S(H) = \{7, 8, 10, 12, 16, 18, 20\}$. Since $8 + 20 = 16 + 12 = 10 + 18$, H is cyclic, hence Weierstrass.

(2) Let $H =< 7, 15, 16, 17, 25, 26, 27 >$. Then $S(H) = \{7, 15, 16, 17, 25, 26, 27\}$. Since $15 + 27 = 16 + 26 = 17 + 25$, H is cyclic, hence Weierstrass.

We also have non-cyclic Weierstrass 7-semigroups.

Fact 9. For integers g and s with $7 \leqq g \leqq s \leqq 12$, let $H_{s,g}$ be a 7-semigroup with

$$\mathbb{N}_0 \backslash H_{s,g} = \{1, \ldots, 6, 8+s-g, \ldots, s+1\}.$$

Then we have the following:
i) There exists a covering $C \longrightarrow \mathbb{P}^1$ of degree 3 with non-ramification point $P \in C$ such that $H(P) = H_{s,g}$. Hence, $H_{s,g}$ is a Weierstrass 7-semigroup (See [2]).
ii) If $(s,g) \neq (9,9), (12,9), (12,12)$, then $H_{s,g}$ is non-cyclic. For example, $H_{11,9} = < 7, 8, 9, 13, 19 >$ and $H_{12,10} =< 7, 8, 9, 19, 20 >$ are non-cyclic Weierstrass 7-semigroups.

Fact 10. Let H be the 7-semigroup $< 7, 9, 10, 11, 12, 13 >$. Then there is a cyclic covering of an elliptic curve of degree 8 with only two ramification points P_1 and P_2, which are totally ramified, such that $H(P_1) = H(P_2) = H$. Hence $< 7, 9, 10, 11, 12, 13 >$ is a non-cyclic Weierstrass 7-semigroup (See [6]).

§3. 7-semigroups of toric type.

For a numerical semigroup H we denote by $M(H)$ the minimal set of generators for H. In this section we are interested in 7-semigroups H with $M(H) = \{7, a_1, a_2, a_3\}$ which satisfy the following condition:

Definition 11. Let H be a numerical semigroup with $\sharp M(H) = m+1$. The semigroup H is said to be *of toric type* if
$\exists\ l$: a positive integer,
$\exists\ S$: a saturated subsemigroup of $\mathbb{Z}^l$ generated by $b_1, \ldots, b_{l+m}$ which generates $\mathbb{Z}^l$ as a group and
$\exists\ g_j$'s $(j = 1, \ldots, l+m)$: monomials in $k[X_0, X_1, \ldots, X_m]$ such that

$$\begin{array}{ccc} \text{Spec } k[H] & \hookrightarrow & \text{Spec } k[S][X_0, X_1, \ldots, X_m] \\ \downarrow & \square & \downarrow \\ \text{Spec } k & \hookrightarrow & \text{Spec } k[Y_1, \ldots, Y_{l+m}] \\ (0) & \longmapsto & \text{the origin} \end{array}$$

where the right vertical map is induced by the k-algebra homomorphism

$$\eta_S : k[Y_1, \ldots, Y_{l+m}] \longrightarrow k[S][X_0, X_1, \ldots, X_m]$$

which sends Y_j to $T^{b_j} - g_j$, that is to say,

$$\begin{array}{ccc} \text{Spec } k[H] & \hookrightarrow & \text{Spec } k[X_0, X_1, \ldots, X_m] \\ \downarrow & \square & \downarrow \\ \text{Spec } k[S] & \hookrightarrow & \text{Spec } k[Y_1, \ldots, Y_{l+m}] \end{array}$$

where the horizontal maps are the embeddings through the generators and the right vertical map is induced by the k-algebra morphism from $k[Y_1, \ldots, Y_{l+m}]$ to $k[X_0, X_1, \ldots, X_m]$ sending Y_j to g_j.

We explain how to find a subsemigroup S of $\mathbb{Z}^l$ as in Definition 11 below.

Remark 12. Let H be a numerical semigroup with $M(H) = \{a_0, a_1, \ldots, a_m\}$.
i) Determine a generating system of relations among $a_0, a_1, \ldots, a_m$, i.e., a set of generators for the ideal of the monomial curve Spec $k[H]$.
ii) Determine a fundamental system of relations among $a_0, a_1, \ldots, a_m$, i.e., a basis of the relation $\mathbb{Z}$-module among $a_0, a_1, \ldots, a_m$.
iii) We construct a subsemigroup S of $\mathbb{Z}^l$ from the fundamental system In this case, S is generated by $l+m$ elements b_j's and generates $\mathbb{Z}^l$ as a group naturally. Moreover, we associate the generators b_j's for S to monomials g_j's in $k[X_0, \ldots, X_m]$ such that we have the fiber products in Definition 11.
iv) The remaining problem is whether the semigroup S is *saturated or not.* We note that S is saturated if and only if the semigroup ring $k[S]$ is normal, i.e., Spec $k[S]$ is an *affine toric variety.* If S is saturated, the numerical semigroup H become *of toric type.*

From now on we treat only 7-semigroups generated by 4 elements.

Lemma 13. *Let H be a 7-semigroup generated by 4 elements, i.e., $M(H) = \{7, a_1, a_2, a_3\}$. Renumbering a_1, a_2 and a_3 it satisfies one of the following:*
(I) $a_1 + a_2 + a_3 \equiv 0\ (7)$,
(II) $a_1 + a_2 \equiv 0\ (7)$,
(III) $2a_1 + a_2 \equiv 0\ (7)$ *and* $2a_2 + a_3 \equiv 0\ (7)$.

We give the construction of a saturated subsemigroup S of $\mathbb{Z}^l$ as in Definition 11 in (I) and some cases of (II).

Case (I) $a_1 + a_2 + a_3 \equiv 0\ (7)$. A fundamental system of relations consists of

$$\frac{a_1 + a_2 + a_3}{7} a_0 = a_1 + a_2 + a_3,\ 2a_1 = \frac{2a_1 - a_2}{7} a_0 + a_2,\ 2a_2 = \frac{2a_2 - a_3}{7} a_0 + a_3.$$

For example, the relation

$$2a_3 = \frac{2a_3 - a_1}{7} a_0 + a_1$$

is derived from the addition of the three relations. The determinant of the matrix consisting of the coefficients of the three relations is

$$\begin{vmatrix} (a_1 + a_2 + a_3)/7 & -1 & -1 \\ -(2a_1 - a_2)/7 & 2 & -1 \\ -(2a_2 - a_3)/7 & 0 & 2 \end{vmatrix} = a_3.$$

A numerical semigroup H with $M(H) = \{a_0, a_1, a_2, a_3\}$ satisfying the above condition is said to be 1-*neat.* Under the above condition we get a saturated subsemigroup S of $\mathbb{Z}^6$ as in Definition 11 from the fundamental system.

Case (II-1) $a_1 + a_2 \equiv 0\ (7)$ and $2a_1 \equiv a_3\ (7)$.
Case (II-1-i) $2a_2 < a_1 + 2a_3$ and $2a_3 < 3a_2$. A generating system for relations consists of

$$\frac{a_1+a_2}{7}a_0 = a_1 + a_2,\ 2a_1 = \frac{2a_1 - a_3}{7}a_0 + a_3,\ 3a_2 = \frac{3a_2 - 2a_3}{7}a_0 + 2a_3,$$

$$3a_3 = \frac{3a_3 - a_2}{7}a_0 + a_2,\ \frac{a_2 + a_3 - a_1}{7}a_0 + a_1 = a_2 + a_3,$$

$$\frac{a_1 + 2a_3 - 2a_2}{7}a_0 + 2a_2 = a_1 + 2a_3.$$

i.e., the kernel of

$$\begin{array}{cccc} \varphi_H : k[X_0, X_1, X_2, X_3] & \longrightarrow & k[t^{a_0}, t^{a_1}, t^{a_2}, t^{a_3}] \\ X_i & \mapsto & t^{a_i} \end{array}$$

is generated by

$$X_0^{\frac{a_1+a_2}{7}} - X_1X_2,\quad X_1^2 - X_0^{\frac{2a_1-a_3}{7}}X_3,\quad X_2^3 - X_0^{\frac{3a_2-2a_3}{7}}X_3,$$

$$X_3 - X_0^{\frac{3a_3-a_2}{7}}X_2,\quad X_0^{\frac{a_2+a_3-a_1}{7}}X_1 - X_2X_3,\quad X_0^{\frac{a_1+2a_3-2a_2}{7}}X_2^2 - X_1X_3^2.$$

A fundamental system of relations is the following:

$$\frac{a_1+a_2}{7}a_0 = a_1 + a_2,\ 2a_1 = \frac{2a_1 - a_3}{7}a_0 + a_3,\ 3a_2 = \frac{3a_2 - 2a_3}{7}a_0 + 2a_3.$$

For example, the addition of the first and second relations

$$\frac{a_1+a_2}{7}a_0 + 2a_1 = \left(a_1 + a_2\right) + \left(\frac{2a_1 - a_3}{7}a_0 + a_3\right)$$

induces the fifth relation. To get a subsemigroup S of $\mathbb{Z}^l$ we divide this case into three cases again.
Case (II-1-i-A) $a_1 + 2a_2 > 3a_3$. We divide the coefficients in the fundamental system of relations into the following:

$$(\alpha_0' + \alpha_0'' + \alpha_0''')a_0 = \alpha_{01}a_1 + \alpha_{02}a_2,\ 2\alpha_{01}a_1 = (\alpha_0' + \alpha_0'')a_0 + \alpha_{13}a_3,$$

$$(2\alpha_{02} + \alpha_2')a_2 = (\alpha_0' + \alpha_0''')a_0 + \alpha_{23}a_3.$$

We associate elements of $\mathbb{Z}^5$ to the components of the above system as follows:

$$\alpha_0'a_0 \mapsto \mathbf{b}_1 = \mathbf{e}_1,\ \alpha_0''a_0 \mapsto \mathbf{b}_2 = \mathbf{e}_2,\ \alpha_0'''a_0 \mapsto \mathbf{b}_3 = \mathbf{e}_3,\ \alpha_{01}a_1 \mapsto \mathbf{b}_4 = \mathbf{e}_4,$$

$$\alpha_2'a_2 \mapsto \mathbf{b}_5 = \mathbf{e}_5,\ \alpha_{02}a_2 \mapsto \mathbf{b}_6 = (1, 1, 1, -1, 0),$$

$$\alpha_{13}a_3 \mapsto \mathbf{b}_7 = (-1,-1,0,2,0),\ \alpha_{23}a_3 \mapsto \mathbf{b}_8 = (1,2,1,-2,1).$$

where $\mathbf{e}_i$ denotes the vector whose i-th component is 1 and j-th component is 0 if $j \neq i$. Let S be the subsemigroup of $\mathbb{Z}^5$ generated by $\mathbf{b}_1, \ldots, \mathbf{b}_8$. We can show that

$$\sum_{i=1}^{8} \mathbb{R}_+ \mathbf{b}_i \cap \mathbb{Z}^5 \subseteq S$$

where $\mathbb{R}_+$ denotes the set of non-negative real numbers. Hence, S is saturated.

Case (II-1-i-B) $a_1 + 2a_2 < 3a_3$. We divide the coefficients in the fundamental system of relations into the following:

$$(\alpha_0' + \alpha_{10} + \alpha_{20})a_0 = \alpha_{01}a_1 + \alpha_{02}a_2,\ 2\alpha_{01}a_1 = \alpha_{10}a_0 + \alpha_{13}a_3,$$

$$(2\alpha_{02} + \alpha_2')a_2 = \alpha_{20}a_0 + \alpha_{23}a_3.$$

We associate elements of $\mathbb{Z}^5$ to the components of the above system as follows:

$$\alpha_0'a_0 \mapsto \mathbf{b}_1 = \mathbf{e}_1,\ \alpha_{10}a_0 \mapsto \mathbf{b}_2 = \mathbf{e}_2,\ \alpha_{20}a_0 \mapsto \mathbf{b}_3 = \mathbf{e}_3,\ \alpha_{01}a_1 \mapsto \mathbf{b}_4 = \mathbf{e}_4,$$

$$\alpha_2'a_2 \mapsto \mathbf{b}_5 = \mathbf{e}_5, \alpha_{02}a_2 \mapsto \mathbf{b}_6 = (1,1,1,-1,0),$$

$$\alpha_{13}a_3 \mapsto \mathbf{b}_7 = (0,-1,0,2,0),\ \alpha_{23}a_3 \mapsto \mathbf{b}_8 = (2,2,1,-2,1).$$

Let S be the subsemigroup of $\mathbb{Z}^5$ generated by $\mathbf{b}_1, \ldots, \mathbf{b}_8$. Then S is saturated.

Case (II-1-i-C) $a_1 + 2a_2 = 3a_3$. In the Case (II-1-i-A) let $\alpha_0' = 0$. We get a subsemigroup S of $\mathbb{Z}^4$ generated by 7 elements. Then S is saturated.

But our method does not work well in the following case.

Case (III-2-i) $2a_1 + a_2 \equiv 0$, $2a_2 + a_3 \equiv 0$, $2a_1 \leqq a_2 + a_3$, $2a_2 > 3a_1$. We have the following generating system of relations

$$\frac{2a_1 + a_2}{7} a_0 = 2a_1 + a_2, \tag{1}$$

$$4a_1 = \frac{4a_1 - a_3}{7} a_0 + a_3, \tag{2}$$

$$2a_2 = \frac{2a_2 - 3a_1}{7} a_0 + 3a_1, \tag{3}$$

$$2a_3 = \frac{2a_3 - a_1}{7} a_0 + a_1, \tag{4}$$

$$\frac{a_2 + a_3 - 2a_1}{7} a_0 + 2a_1 = a_2 + a_3, \tag{5}$$

$$\frac{a_1 + a_3 - a_2}{7} a_0 + a_2 = a_1 + a_3. \tag{6}$$

The three equations (1), (2) and (6) in the generating system of relations form a fundamental system. In fact,

$$(1)+(2)=(5),\ {}^t(1)+{}^t(2)+(6)=(3) \text{ and } {}^t(1)+{}^t(2)+{}^t(6)=(4).$$

We divide the coefficients in the fundamental system of relations into the following:

$$(\alpha_{10}+\alpha_{20}+\alpha'_0)a_0=\alpha_{01}a_1+\alpha'_2a_2,\ (\alpha_{01}+\alpha'_1+\alpha_{31})a_1=\alpha_{10}a_0+\alpha_{13}a_3,$$

$$\alpha'_0a_0+\alpha'_2a_2=\alpha'_1a_1+\alpha_{13}a_3.$$

We associate elements of $\mathbb{Z}^5$ to the components of the above system as follows:

$$\alpha_{10}a_0\mapsto \mathbf{b}_1=\mathbf{e}_1,\ \alpha_{20}a_0\mapsto \mathbf{b}_2=\mathbf{e}_2,\ \alpha'_0a_0\mapsto \mathbf{b}_3=\mathbf{e}_3,\ \alpha_{01}a_1\mapsto \mathbf{b}_4=\mathbf{e}_4,$$

$$\alpha'_1a_1\mapsto \mathbf{b}_5=\mathbf{e}_5,\ \alpha'_2a_2\mapsto \mathbf{b}_6=(1,1,1,-1,0),$$

$$\alpha_{31}a_1\mapsto \mathbf{b}_7=(2,1,2,-2,-2),\ \alpha_{13}a_3\mapsto \mathbf{b}_8=(1,1,2,-1,-1).$$

Let S be the subsemigroup of $\mathbb{Z}^5$ generated by $\mathbf{b}_1,\ldots,\mathbf{b}_8$. Then S is not saturated. In fact,

$$2(1,1,1,-1,-1)=(2,2,2,-2,-2)=\mathbf{b}_2+\mathbf{b}_7\in S,$$

but

$$(1,1,1,-1,-1)\notin S \text{ and } (1,1,1,-1,-1)\in\mathbb{Z}^5.$$

Hence, Spec $k[S]$ is *not a toric variety.*

To check whether a 7-semigroup generated by 4 elements is of toric type we divide them into the 31 cases in the following table. But this problem is still open in the last three cases. The right-hand side of column in the table means the dimension of the affine toric variety which is constructed from a numerical semigroup of given type in our way.

	Condition	Toric	dim
I	$a_1 + a_2 + a_3 \equiv 0$	○	6
II-1-i-A	$a_1 + a_2 \equiv 0,\, 2a_1 \equiv a_3,\, 2a_2 < a_1 + 2a_3,\, 2a_3 < 3a_2,\, a_1 + 2a_2 > 3a_3$	○	5
II-1-i-B	$a_1 + a_2 \equiv 0,\, 2a_1 \equiv a_3,\, 2a_2 < a_1 + 2a_3,\, 2a_3 < 3a_2,\, a_1 + 2a_2 < 3a_3$	○	5
II-1-i-C	$a_1 + a_2 \equiv 0,\, 2a_1 \equiv a_3,\, 2a_2 < a_1 + 2a_3,\, 2a_3 < 3a_2,\, a_1 + 2a_2 = 3a_3$	○	4
II-1-ii-A	$a_1 + a_2 \equiv 0,\, 2a_1 \equiv a_3,\, 2a_3 > 3a_2,\, 4a_2 > a_1 + a_3$	○	6
II-1-ii-B	$a_1 + a_2 \equiv 0,\, 2a_1 \equiv a_3,\, 2a_3 > 3a_2,\, 4a_2 = a_1 + a_3$	○	5
II-1-ii-C	$a_1 + a_2 \equiv 0,\, 2a_1 \equiv a_3,\, 2a_3 > 3a_2,\, 4a_2 < a_1 + a_3$	○	5
II-1-iii-A	$a_1 + a_2 \equiv 0,\, 2a_1 \equiv a_3,\, 2a_2 > a_1 + 2a_3$	○	6
II-1-iii-B	$a_1 + a_2 \equiv 0,\, 2a_1 \equiv a_3,\, 2a_2 = a_1 + 2a_3$	○	5
II-1-iv	$a_1 + a_2 \equiv 0,\, 2a_1 \equiv a_3,\, 2a_3 = 3a_2$	○	4
II-2-i-A	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 > 2a_1 + a_3$	○	6
II-2-i-B	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 = 2a_1 + a_3$	○	5
II-2-ii-A	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 > a_2 + a_3,\, 3a_2 > a_1 + a_3$	○	7
II-2-ii-B	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 > a_2 + a_3,\, 3a_2 = a_1 + a_3$	○	6
II-2-ii-C	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 > a_2 + a_3,\, 3a_2 < a_1 + a_3,\, a_1 + 2a_2 > 2a_3$	○	6
II-2-ii-D	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 > a_2 + a_3,\, 3a_2 < a_1 + a_3,\, a_1 + 2a_2 = 2a_3$	○	5
II-2-ii-E	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 > a_2 + a_3,\, 3a_2 < a_1 + a_3,\, a_1 + 2a_2 < 2a_3$	○	6
II-2-iii-A	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 < a_2 + a_3,\, 3a_2 \geqq a_1 + a_3,\, a_1 + 2a_2 > 2a_3$	○	6
II-2-iii-B	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 < a_2 + a_3,\, 3a_2 \geqq a_1 + a_3,\, a_1 + 2a_2 = 2a_3$	○	5
II-2-iii-C	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 < a_2 + a_3,\, 3a_2 > a_1 + a_3,\, a_1 + 2a_2 < 2a_3$	○	6
II-2-iii-D	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 < a_2 + a_3,\, 3a_2 = a_1 + a_3,\, a_1 + 2a_2 < 2a_3$	○	5
II-2-iii-E	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 < a_2 + a_3,\, 3a_2 < a_1 + a_3,\, 2a_1 + 3a_2 < 2a_3$	○	5
II-2-iii-F	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 < a_2 + a_3,\, 3a_2 < a_1 + a_3,\, 2a_1 + 3a_2 = 2a_3$	○	4
II-2-iii-G	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 < a_2 + a_3,\, 3a_2 < a_1 + a_3,\, 2a_1 + 3a_2 > 2a_3$	○	5
II-2-iv-A	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 = a_2 + a_3,\, 3a_2 > a_1 + a_3$	○	6
II-2-iv-B	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 = a_2 + a_3,\, 3a_2 < a_1 + a_3$	○	6
II-2-iv-C	$a_1 + a_2 \equiv 0,\, 3a_1 \equiv a_3,\, 2a_2 < 2a_1 + a_3,\, 2a_1 = a_2 + a_3,\, 3a_2 = a_1 + a_3$	○	5
III-1	$2a_1 + a_2 \equiv 0,\, 2a_2 + a_3 \equiv 0,\, 2a_1 > a_2 + a_3$	○	6
III-2-i	$2a_1 + a_2 \equiv 0,\, 2a_2 + a_3 \equiv 0,\, 2a_1 \leqq a_2 + a_3,\, 2a_2 > 3a_1$	?	(5)
III-2-ii	$2a_1 + a_2 \equiv 0,\, 2a_2 + a_3 \equiv 0,\, 2a_1 \leqq a_2 + a_3,\, 2a_2 < 3a_1$	?	(5)
III-2-iii	$2a_1 + a_2 \equiv 0,\, 2a_2 + a_3 \equiv 0,\, 2a_1 \leqq a_2 + a_3,\, 2a_2 = 3a_1$	?	(4)

References

[1] R.O. Buchweitz, On Zariski's criterion for equisingularity and non-smoothable monomial curves, preprint 113, University of Hannover, 1980.

[2] S.J. Kim, On the existence of Weierstrass gap sequences on trigonal curves, J. Pure Appl. Algebra 63 (1990) 171-180.

[3] S.J. Kim and J. Komeda, Numerical semigroups which cannot be realized as semigroups of Galois Weierstrass points, Arch. Math. 76 (2001) 265-273.

[4] J. Komeda, On Weierstrass points whose first non-gaps are four, J. Reine Angew. Math. 341 (1983) 68–86.

[5] J. Komeda, On the existence of Weierstrass points whose first non-gaps are five, Manuscripta Math. 76 (1992) 193–211.

[6] J. Komeda, Cyclic coverings of an elliptic curve with two branch points and the gap sequences at the ramification points, Acta Arithmetica LXXXI (1997) 275-297.

[7] J. Komeda, Non-Weierstrass numerical semigroups, Semigroup Forum 57 (1998) 157-185.

[8] C. Maclachlan, Weierstrass points on compact Riemann surfaces, J. London Math. Soc. 3 (1971) 722–724.

[9] I. Morrison and H. Pinkham, Galois Weierstrass points and Hurwitz characters, Ann. Math. 124 (1986) 591-625.

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